Monochromatic and Heterochromatic Subgraph Problems in a Randomly Colored Graph[1]

Xueliang Li and Jie Zheng

Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China

Email: x.li@eyou.com, jzheng@eyou.com

Abstract. Let K_n be the complete graph with n vertices and $c_1, c_2, \cdots, c_r$ be r different colors. Suppose we randomly and uniformly color the edges of K_n in $c_1, c_2, \cdots, c_r$. Then we get a random graph, denoted by $\mathcal{K}_n^r$. In the paper, we investigate the asymptotic properties of several kinds of monochromatic and heterochromatic subgraphs in $\mathcal{K}_n^r$. Accurate threshold functions in some cases are also obtained.

Keywords: monochromatic, heterochromatic, threshold function

AMS Classification: 05C80, 05C15

1 Introduction

The study of random graphs was begun by P. Erdös and A. Rényi in the 1960s [7–9] and now has a comprehensive literature [3, 6].

The most frequently encountered probabilistic model of random graph is $\mathcal{G}_{n,p(n)}$, where $0 \leq p(n) \leq 1$. It consists of all graphs with vertex set $V = \{1, 2, \cdots, n\}$ in which the edges are chosen independently and with probability $p(n)$. As $p(n)$ goes from zero to one the random graph $\mathcal{G}_{n,p(n)}$ evolves from empty to full.

[1]Supported by NSFC, PCSIRT and the "973" program.

P. Erdös and A. Rényi discovered that for many natural properties A of graphs there was a narrow range in which $Pr[\mathcal{G}_{n,p(n)}$ has property A] moves from near zero to near one. So we introduce the following important definition ([5], page 14).

Definition 1.1 *A function $p(n)$ is a threshold function for property A if the following two conditions are satisfied:*

1. *If $p'(n) \ll p(n)$, then $\lim_{n\to\infty} Pr[\mathcal{G}_{n,p'(n)}$ has property A$] = 0$.*

2. *If $p'(n) \gg p(n)$, then $\lim_{n\to\infty} Pr[\mathcal{G}_{n,p'(n)}$ has property A$] = 1$.*

In general, if $Pr[\mathcal{G}_{n,p(n)}$ has property A$] \to 0$, we say *almost no $\mathcal{G}_{n,p(n)}$* has property A. Conversely, if $Pr[\mathcal{G}_{n,p(n)}$ has property A$] \to 1$, we say *almost every $\mathcal{G}_{n,p(n)}$* has property A.

In this article, we introduce the following probabilistic model of random graphs. Let K_n be the complete graph with vertex set $V = \{1, 2, \cdots, n\}$ and $c_1, c_2, \cdots, c_r$ be $r = r(n)$ different colors. We now send $c_1, c_2, \cdots, c_r$ to the edges of K_n randomly and equiprobably, which means each edge is colored in $c_i (1 \le i \le r)$ with probability $\frac{1}{r}$. Thus we get a random graph $\mathcal{K}_n^r$. The probability space $(\Omega, \mathcal{F}, \mathcal{P})$ of $\mathcal{K}_n^r$ has a simple form: Ω has $r^{\binom{n}{2}}$ elements and each one has probability $\frac{1}{r^{\binom{n}{2}}}$ to appear.

The subgraph of $\mathcal{K}_n^r$ with vertices $1, 2, \cdots, n$ and the edges that have color c_i is denote by $\mathcal{G}_i$. Obviously, it is just the random graph $\mathcal{G}_{n,p(n)}$ ([3], page 34), where $p(n) = \frac{1}{r}$.

Matching, clique and tree are three kinds of important subgraphs. As to their definitions, please refer to [2]. A k-matching is a matching of k independent edges. A k-clique is a clique of k vertices. Similar, a k-tree is a tree of k vertices. In a k-matching (k-clique, k-tree), if all of the edges are in a same color, we call it a monochromatic k-matching (k-clique, k-tree); On the other hand, if any two of edges are of different colors, we call it a heterochromatic k-matching (k-clique, k-tree).

Having a monochromatic k-matching, k-clique or k-tree or a heterochromatic k-matching, k-clique or k-tree are all properties of $\mathcal{K}_n^r$. We want to investigate these properties and obtain the threshold functions for them.

Two properties will be especially demonstrated: monochromatic k-matching and heterochromatic k-matching. For the others, the methods are similar and we list the results in Section 4.

2 Monochromatic k-Matchings in $\mathcal{K}_n^r$

Let k be an integer. Obviously, in $\mathcal{K}_n^r$, there are altogether

$$q = \frac{\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2k+2}{2}}{k!}$$

sets of k independent edges. Arrange them in an order and the i-th one is denoted by M_i.

Let A_i be the event that the edges in M_i are monochromatic and X_i be the indicator variable for A_i. That is,

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ happens,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then the random variable

$$X = X_1 + X_2 + \cdots + X_q$$

denotes the number of monochromatic k-matchings in $\mathcal{K}_n^r$.

For each $1 \le i \le q$,

$$E(X_i) = Pr[X_i = 1] = \frac{r}{r^k}.$$

From the linear of the expectation [1],

$$\begin{aligned} E(X) &= E(X_1 + X_2 + \cdots + X_q) \\ &= \frac{r}{r^k} q \\ &= \frac{n!}{(n-2k)!2^k k! r^{k-1}}. \end{aligned} \tag{2.2}$$

By careful calculation, the following assertions (*) for (2.2) are true, which will be used later:

1. If r is fixed, then for every $1 \leq k \leq \frac{n}{2}$, $E(X) \to \infty$.

2. If k is fixed and $r \ll (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}$, then $E(X) \to \infty$;

3. If k is fixed and $r \gg (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}$, then $E(X) \to 0$;

4. If k is fixed and $r = c^{(0)}(\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}$, where $c^{(0)} > 0$ is a constant, then $E(X) \to \frac{1}{(c^{(0)})^{k-1}}$.

Though k and r can be both functions of n, if they are both variables, the situation becomes very complicated. So we illustrate monochromatic k-matching problem from three aspects: r is fixed, k is fixed and $k = \lfloor \frac{n}{2} \rfloor$. The last case is the perfect matching case or the nearly perfect matching case. Since we focus on the asymptotic properties, we will not distinguish $\lfloor \frac{n}{2} \rfloor$ from $\frac{n}{2}$. That is, we always suppose n is an even.

2.1 r is fixed

Assertion (*) 1 says that $E(X) \to \infty$ for every $1 \leq k \leq \frac{n}{2}$ if r is fixed. We certainly expect that $Pr[X > 0] \to 1$ holds. In fact, it does.

Theorem 2.1 *If $r \geq 1$ is fixed, then almost every $\mathcal{K}_n^r$ has a monochromatic k-matching for any $1 \leq k \leq \frac{n}{2}$.*

Proof. We have mentioned in Section 1 that the subgraph $\mathcal{G}_i$ of $\mathcal{K}_n^r$ is actually the random graph $\mathcal{G}_{n,p(n)}$, where $p(n) = \frac{1}{r}$. There is a result saying that the threshold function for $\mathcal{G}_{n,p}$ has a perfect matching is $\frac{logn}{n}$ ([6], page 85). If r is fixed, then $\frac{1}{r} \gg \frac{logn}{n}$, which implies that almost every $\mathcal{G}_i$ has a perfect matching. Then almost every $\mathcal{K}_n^r$ has a monochromatic k-matching for every $1 \leq k \leq \frac{n}{2}$. ∎

2.2 k is fixed

In this case, we prove the following theorem.

Theorem 2.2 *If k is fixed ($k = 1$ is a trivial case so suppose $k \geq 2$), then*

$$\lim_{n\to\infty} Pr[X > 0] = \begin{cases} 0 & \text{if } r \gg (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}, \\ 1 & \text{if } r \ll (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}. \end{cases} \tag{2.3}$$

That is to say, $(\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}$ is the threshold function for the property that $\mathcal{K}_n^r$ has a monochromatic k-matching.

Proof. From *Markov's inequality* [4]

$$Pr[X > 0] \leq E(X)$$

and assertion (*) 3, we have

$$Pr[X > 0] \to 0 \text{ if } r \gg (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}.$$

For the other half, we estimate $\frac{\Delta}{(E(X))^2}$, where $\Delta = \sum_{i\sim j} Pr[A_i \cap A_j]$. $A_i(A_j)$ denotes the event that the edges in $M_i(M_j)$ are monochromatic and $i \sim j$ means the ordered pair of A_i and A_j that are not independent from each other.

Our goal is to prove that if $r \ll (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}$, then $\frac{\Delta}{(E(X))^2} \to 0$. Because

$$\begin{aligned}
\Delta &= \sum_{i\sim j} Pr[A_i \cap A_j] \\
&= \sum_{s=1}^{k-1} \sum_{(i,j)_s} \frac{r}{r^{2k-s}} \\
&\quad \text{(where } (i,j)_s \text{ means the ordered pair of } M_i \text{ and } M_j \text{ that have } s \text{ common edges)} \\
&\leq \sum_{s=1}^{k-1} \frac{\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2(s-1)}{2}}{s!} \left(\frac{\binom{n-2s}{2}\binom{n-2s-2}{2}\cdots\binom{n-2k+2}{2}}{(k-s)!}\right)^2 \frac{1}{r^{2k-s-1}} \\
&= \frac{n!}{2^{2k}(n-2k)!(n-2k)!r^{2k-1}} \sum_{s=1}^{k-1} \frac{(n-2s)!2^s r^s}{s!(k-s)!(k-s)!},
\end{aligned}$$

then we have

$$\frac{\Delta}{(E(X))^2} \leq \frac{k!k!}{n!} \sum_{s=1}^{k-1} \frac{(n-2s)!2^s r^{s-1}}{s!(k-s)!(k-s)!}. \tag{2.4}$$

If $r \ll (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}} \sim (\frac{1}{2^k k!})^{\frac{1}{k-1}} n^{\frac{2k}{k-1}}$, then there are 3 possible cases: (i) $r \ll n^2$, (ii) $r = c^{(1)} n^2$, where $c^{(1)} > 0$ is a constant and (iii) $n^2 \ll r \ll (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}$.

In case (i),

$$\sum_{s=1}^{k-1} \frac{(n-2s)!2^s r^s}{s!(k-s)!(k-s)!} = (1+\circ(1)) \frac{2(n-2)!}{(k-1)!(k-1)!}. \tag{2.5}$$

Then submit (2.5) to (2.4), we get

$$\frac{\Delta}{(E(X))^2} \leq 2(1+\circ(1)) \frac{k^2}{n(n-1)} \to 0. \tag{2.6}$$

In case (ii)

$$\sum_{s=1}^{k-1} \frac{(n-2s)!2^s r^s}{s!(k-s)!(k-s)!} = c^{(2)} \frac{2(n-2)!}{(k-1)!(k-1)!}, \tag{2.7}$$

where $c^{(2)}$ is a sufficiently large constant.

Then submit (2.7) to (2.4), we get

$$\frac{\Delta}{(E(X))^2} \leq 2c^{(2)} \frac{k^2}{n(n-1)} \to 0. \tag{2.8}$$

In case (iii)

$$\sum_{s=1}^{k-1} \frac{(n-2s)!2^s r^s}{s!(k-s)!(k-s)!} = (1+\circ(1)) \frac{(n-2k+2)!2^{k-1} r^{k-2}}{(k-1)!}. \tag{2.9}$$

Then submit (2.9) to (2.4), we get

$$\frac{\Delta}{(E(X))^2} \leq c^{(3)} \frac{n^{\frac{2k(k-2)}{k-1}}}{n^{2k-2}} \to 0, \tag{2.10}$$

where $c^{(3)}$ is a sufficiently large constant.

Summarizing (2.6) (2.8) and (2.10), we end the proof of $\frac{\Delta}{(E(X))^2} \to 0$ with the condition $r \ll (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}$.

A corollary of the *Chebyshev's inequality* [4] asserts that if $E(X) \to \infty$ and $\Delta = \circ((E(X))^2)$, then almost surely $X > 0$([1],page 46). So from assertion (*) 2 and the above discuss, we obtain

$$Pr[X > 0] \to 1 \text{ if } r \ll (\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}.$$

From the definition of the threshold function (Definition 1.1), we can say that $(\frac{n!}{(n-2k)!2^k k!})^{\frac{1}{k-1}}$ is the threshold function for the property that $\mathcal{K}_n^r$ has a monochromatic k-matching. ∎

2.3 $k = \frac{n}{2}$

When $k = \frac{n}{2}$, a monochromatic k-matching is a monochromatic perfect matching.

Replace k with $\frac{n}{2}$ in (2.2), we have

$$E(X) = \frac{n!}{(\frac{n}{2})!2^{\frac{n}{2}} r^{\frac{n}{2}-1}}. \tag{2.11}$$

By calculation of (2.11), we get $E(X) \to 0$ if $r \geq \frac{n}{c^{(4)}}$, where $c^{(4)} < e$ is a constant; $E(X) \to \infty$ if $r \leq \frac{n}{e}$.

The following assertion is true as a direct corollary of Markov's inequality and the threshold function for the property that $\mathcal{G}_{n,p}$ having a perfect matching ([6], page 85). Here we omit its proof.

Theorem 2.3 *If $r \geq \frac{n}{c^{(4)}}$, where $c^{(4)} < e$ is a constant, then almost no $\mathcal{K}_n^r$ has a monochromatic perfect matching. On the other hand, if $r \leq \frac{n}{logn + c^{(5)}(n)}$, where $c^{(5)}(n) \to \infty$, then almost every $\mathcal{K}_n^r$ has a monochromatic perfect matching.*

3 Heterochromatic k-Matchings in $\mathcal{K}_n^r$

Following the symbols in the previous section, let B_i be the event that the edges in M_i are heterochromatic and Y_i be the indicator variable for the

event B_i. That is,

$$Y_i = \begin{cases} 1 & \text{if } B_i \text{ happens,} \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

Then for each $1 \leq i \leq q$,

$$Pr[Y_i = 1] = \frac{\binom{r}{k} k!}{r^k}.$$

Then the random variable

$$Y = Y_1 + Y_2 + \cdots + Y_q$$

denotes the number of heterochromatic k-matchings in $\mathcal{K}_n^r$.

From the linear of the expectation [1],

$$\begin{aligned} E(Y) &= E(Y_1 + Y_2 + \cdots + Y_q) \\ &= \frac{r}{r^k} q \\ &= \frac{n!}{(n-2k)! 2^k k!} \frac{r!}{(r-k)! r^k}. \end{aligned} \quad (3.2)$$

Since $r \geq k$ is a necessary condition in the heterochromatic k-matching problem, we have the following assertion for $E(Y)$ by calculation of (3.2).

Lemma 3.1 *For every $1 \leq k \leq n^{1-\epsilon}$ and $r \geq k$, $E(Y) \to \infty$, where $0 < \epsilon < 1$ is a constant that can be arbitrarily small.*

The main result of this section is the following theorem:

Theorem 3.2 *If $1 \leq k \leq n^{1-\epsilon}$ and $r \geq k$, where $0 < \epsilon < 1$ is a constant that can be arbitrarily small, then almost every $\mathcal{K}_n^r$ contains a heterochromatic k-matching.*

Proof. Similar to Theorem 2.2, for heterochromatic k-matchings, the following estimate is for $\Delta' = \sum_{i \sim j} Pr[B_i \cap B_j]$.

$$\begin{aligned}
\Delta' &= \sum_{i\sim j} Pr[B_i \cap B_j] \\
&= \sum_{s=1}^{k-1} \sum_{(i,j)_s} \frac{\binom{r}{k} k! \binom{r-s}{k-s}(k-s)!}{r^{2k-s}} \\
&\quad \text{(where } (i,j)_s \text{ means the ordered pair of } M_i \text{ and } M_j \text{ that have } s \text{ common edges)} \\
&\leq \sum_{s=1}^{k-1} \frac{\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2(s-1)}{2}}{s!} \Big(\frac{\binom{n-2s}{2}\binom{n-2(s+1)}{2}\cdots\binom{n-2(k-1)}{2}}{(k-s)!}\Big)^2 \frac{\binom{r}{k} k! \binom{r-s}{k-s}(k-s)!}{r^{2k-s}} \\
&= \frac{n!r!}{2^{2k}(n-2k)!(n-2k)!(r-k)!(r-k)!r^{2k}} \sum_{s=1}^{k-1} \frac{(n-2s)!(r-s)!2^s r^s}{s!(k-s)!(k-s)!}.
\end{aligned}$$

Then

$$\frac{\Delta'}{(E(Y))^2} \leq \frac{k!k!}{n!r!} \sum_{s=1}^{k-1} \frac{(n-2s)!(r-s)!(2r)^s}{s!(k-s)!(k-s)!}. \tag{3.3}$$

By careful calculation of (3.3), we get if $k \ll n$, then

$$\begin{aligned}
\frac{\Delta'}{(E(Y))^2} &\leq \frac{k!k!}{n!r!}(1+\circ(1))\frac{2r!(n-2)!}{(k-1)!(k-1)!} \\
&= (1+\circ(1))\frac{k^2}{n(n-1)} \to 0.
\end{aligned} \tag{3.4}$$

From (3.4), Lemma 3.1 and the assertion that if $E(Y) \to \infty$ and $\Delta' = \circ((E(Y))^2)$, then almost surely $Y > 0$([1], page 46), we have

$$Pr[Y > 0] \to 1,$$

which finishes the proof. ∎

Remark 3.3 *As a corollary of Theorem 3.2, if one of k and $r(\geq k)$ is fixed, then almost every $\mathcal{K}_n^r$ has a heterochromatic k-matching. The only left case that we can not deal with is that $k = c^{(6)}n$, where $0 < c^{(6)} \leq \frac{1}{2}$ is a constant.*

4 Results on Other Subgraphs

Completely similar to Section 2 and Section 3, we can study monochromatic k-clique, k-tree and heterochromatic k-clique, k-tree in $\mathcal{K}_n^r$. We list our results here.

Theorem 4.1 *If r is fixed, then*

$$\lim_{n\to\infty} Pr[\mathcal{K}_n^r \textit{ contains a monochromatic } k\textit{-clique}] = \begin{cases} 0 & \textit{if } k \geq 2log_r n, \\ 1 & \textit{if } k \leq \frac{log_r n}{1.704\times 10^9}. \end{cases}$$

Theorem 4.2 *If k is fixed, then*

$$\lim_{n\to\infty} Pr[\mathcal{K}_n^r \textit{ contains a monochromatic } k\textit{-clique}] = \begin{cases} 0 & \textit{if } r \gg n^{\frac{k}{\binom{k}{2}-1}}, \\ 1 & \textit{if } r \leq (\frac{1}{2k!})^{\frac{1}{\binom{k}{2}-1}} n^{\frac{k}{\binom{k}{2}-1}}. \end{cases}$$

That is to say, $n^{\frac{k}{\binom{k}{2}-1}}$ is the threshold function for the property that $\mathcal{K}_n^r$ has a monochromatic k-clique.

Theorem 4.3 *If $r \geq n^{4+\epsilon}$, where $\epsilon > 0$ is a constant that can be arbitrarily small, then for every $k \leq n$, there almost surely exists a heterochromatic k-clique in $\mathcal{K}_n^r$.*

Theorem 4.4 *If k is fixed, then*

$$\lim_{n\to\infty} Pr[\mathcal{K}_n^r \textit{ contains a monochromatic } k\textit{-tree}] = \begin{cases} 0 & \textit{if } r \gg k\binom{n}{k}^{\frac{1}{k-2}}, \\ 1 & \textit{if } r \leq \frac{k}{n}\binom{n}{k}^{\frac{1}{k-2}}. \end{cases}$$

Theorem 4.5 *If $r \geq c^{(7)}n$, where $c^{(7)} > 1$ is a constant, then almost no $\mathcal{K}_n^r$ contains a monochromatic spanning tree.*

Theorem 4.6 *If r is fixed, then almost every $\mathcal{K}_n^r$ contains a monochromatic k-tree for any $2 \leq k \leq n$.*

Theorem 4.7 *If $2 \leq k \leq logn$ and $r \geq k-1$, then almost every $\mathcal{K}_n^r$ contains a heterochromatic k-tree.*

References

[1] N. Alon and J. Spencer, *The Probabilistic Method, 2nd ed.,* John Wiley & Sons, Inc. 2000.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* The Macmillan Press LTD., 1976.

[3] B. Bollobás, *Random Graphs, 2nd ed.,* Cambridge University Press, 2001.

[4] P. Erdös and J. Spencer, *Probabilistic Methods in Combinatorics,* Academic Press, 1974.

[5] J. Spencer, *The Strange Logic of Random Graphs,* Springer, 2001.

[6] S. Janson, T. Łuczak and A. Rucinski, *Random Graphs*, John Wiley & Sons, Inc., 2000.

[7] P. Erdös and A. Rényi, On random graphs I, *Publ. Math. Debrecen* **6**, 290-297.

[8] P. Erdös and A. Rényi, On the evolution of random graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **5**, 17-61.

[9] P. Erdös and A. Rényi, On the evolution of random graphs, *Bull. Inst. Int. Statist. Tokyo* **38**, 343-347.